

MIXING TRANSFORMATIONS IN QUANTUM FIELD THEORY AND NEUTRINO OSCILLATIONS

M.Blasone^{†*1}, P.A.Henning^{*2} and G.Vitiello^{†3}

[†]*Dipartimento di Fisica dell'Università
and INFN, Gruppo Collegato, Salerno
I-84100 Salerno, Italy.*

^{*}*Theoretical Physics, Gesellschaft für Schwerionenforschung GSI
P.O.Box 110552, D-64220 Darmstadt, Germany*

Abstract

Field mixing transformations are studied in quantum field theory and the generator of the transformations is found to induce an $SU(2)$ coherent structure in the vacuum state, both for bosons and for fermions. The Fock space for mixed fields is unitarily inequivalent to the Fock space of the unmixed fields in the infinite volume limit. We study neutrino mixing and oscillations and find that the oscillation amplitude is depressed by a factor which is momentum and mass dependent. The usual formula is recovered in the relativistic limit. Phenomenological features of the modified oscillation formula are discussed. Finally, preliminary results of the Green's function formalism are presented.

¹blasone@vaxsa.csied.unisa.it

²phenning@tpri6c.gsi.de

³vitiello@vaxsa.csied.unisa.it

1 Introduction

We report about recent results in the study of mixing transformations in quantum field theory (QFT) [1, 2], with special attention to the case of neutrino mixing. There are indeed unexpected features in field mixing transformations which, as we will show below, find their origin in the same structure of QFT, in particular in the existence in QFT of infinitely many inequivalent representations of the canonical commutation relations [3,4]. We find that the generator of mixing transformations induces a non trivial structure in the vacuum which turns out to be a coherent state, both for bosons and for fermions. The Fock space for mixed fields appears to be unitarily inequivalent to the Fock space of the original (unmixed) fields in the infinite volume limit.

The question arises if such a new structure leads to any possibly testable effect. For such a purpose, we investigate in detail neutrino mixing and oscillations [5,6] and a new oscillation formula, different from the usual one, is obtained. A correction on the oscillation amplitude is found which turns out to be momentum and mass dependent. In the relativistic limit the usual formula is recovered; this is in general agreement with other studies of neutrino oscillations in the non relativistic region [7].

In Sec.2 we consider boson field mixing as well as fermion field mixing. In Sec.3 we study the neutrino mixing and oscillations and comment upon phenomenological implications. In Sec.4 we present preliminary results on the Green's functions approach to mass mixing.

2 Mixing transformations in quantum field theory

Let us first discuss the case of charged boson mixing. Consider two charged boson fields $\phi_i(x)$, $i = 1, 2$ and their conjugate momenta $\pi_i(x) = \partial_0 \phi_i^\dagger(x)$, satisfying the usual commutation relations with non-zero commutators given by:

$$[\phi_i(x), \pi_i(y)]_{t=t'} = [\phi_i^\dagger(x), \pi_i^\dagger(y)]_{t=t'} = i\delta^3(\bar{x} - \bar{y}) , \quad [a_{k,i}, a_{p,i}^\dagger] = [b_{k,i}, b_{p,i}^\dagger] = \delta^3(\bar{k} - \bar{p}) \quad (1)$$

$$\phi_i(x) = \int \frac{d^3\bar{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_i}} \left(a_{k,i} e^{-ik \cdot x} + b_{k,i}^\dagger e^{ik \cdot x} \right) \quad (2)$$

$$\pi_i(x) = \int \frac{d^3\bar{k}}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega_i}{2}} i \left(a_{k,i}^\dagger e^{ik \cdot x} - b_{k,i} e^{-ik \cdot x} \right) \quad (3)$$

with $k \cdot x = \omega t - \bar{k} \cdot \bar{x}$. Here and in the following we omit vector notation in the indices. The mixing relations are:

$$\begin{aligned}\phi_A(x) &= \phi_1(x) \cos \theta + \phi_2(x) \sin \theta \\ \phi_B(x) &= -\phi_1(x) \sin \theta + \phi_2(x) \cos \theta\end{aligned}\quad (4)$$

and h.c. and similar ones for π_A and π_B . We put them into the form:

$$\begin{aligned}\phi_A(x) &= G^{-1}(\theta, t) \phi_1(x) G(\theta, t) \\ \phi_B(x) &= G^{-1}(\theta, t) \phi_2(x) G(\theta, t)\end{aligned}\quad (5)$$

and similar ones for π_A and π_B ; $G(\theta, t)$ is given by

$$G(\theta, t) = \exp \left[-i \theta \int d^3 \bar{x} \left(\pi_1(x) \phi_2(x) - \pi_2^\dagger(x) \phi_1^\dagger(x) - \pi_2(x) \phi_1(x) + \pi_1^\dagger(x) \phi_2^\dagger(x) \right) \right] \quad (6)$$

and is (at finite volume) an unitary operator: $G^{-1}(\theta, t) = G(-\theta, t) = G^\dagger(\theta, t)$. By introducing the operator $S_+ \equiv -i \int d^3 \bar{x} (\pi_1(x) \phi_2(x) - \pi_2^\dagger(x) \phi_1^\dagger(x))$ (and $S_- = (S_+)^\dagger$), $G(\theta, t)$ can be written as

$$G(\theta, t) = \exp[\theta(S_+ - S_-)] . \quad (7)$$

It is easy to verify that, introducing S_3 and the total charge S_0 as follows

$$S_3 \equiv \frac{-i}{2} \int d^3 \bar{x} \left(\pi_1(x) \phi_1(x) - \pi_2(x) \phi_2(x) + \pi_2^\dagger(x) \phi_2^\dagger(x) - \pi_1^\dagger(x) \phi_1^\dagger(x) \right) \quad (8)$$

$$S_0 = \frac{Q}{2} \equiv \frac{-i}{2} \int d^3 \bar{x} \left(\pi_1(x) \phi_1(x) - \pi_1^\dagger(x) \phi_1^\dagger(x) + \pi_2(x) \phi_2(x) - \pi_2^\dagger(x) \phi_2^\dagger(x) \right) \quad (9)$$

the $su(2)$ algebra is closed: $[S_+, S_-] = 2S_3$, $[S_3, S_\pm] = \pm S_\pm$, $[S_0, S_3] = [S_0, S_\pm] = 0$. We can expand S_+ and S_- as follows:

$$S_+ = \int d^3 \bar{k} \left(U_k(t) a_{k,1}^\dagger a_{k,2} - V_k^*(t) b_{-k,1} a_{k,2} + V_k(t) a_{k,1}^\dagger b_{-k,2}^\dagger - U_k^*(t) b_{-k,1} b_{-k,2}^\dagger \right) \quad (10)$$

$$S_- = \int d^3 \bar{k} \left(U_k^*(t) a_{k,2}^\dagger a_{k,1} - V_k(t) a_{k,2}^\dagger b_{-k,1}^\dagger + V_k^*(t) b_{-k,2} a_{k,1} - U_k(t) b_{-k,2} b_{-k,1}^\dagger \right) \quad (11)$$

with $U_k(t) \equiv |U_k| e^{i(\omega_1 - \omega_2)t}$, $V_k(t) \equiv |V_k| e^{i(\omega_1 + \omega_2)t}$ and

$$|U_k| \equiv \frac{1}{2} \left(\sqrt{\frac{\omega_1}{\omega_2}} + \sqrt{\frac{\omega_2}{\omega_1}} \right) , \quad |V_k| \equiv \frac{1}{2} \left(\sqrt{\frac{\omega_1}{\omega_2}} - \sqrt{\frac{\omega_2}{\omega_1}} \right) . \quad (12)$$

Since $|U_k|^2 - |V_k|^2 = 1$, one can put $|U_k| \equiv \cosh \sigma_k$, $|V_k| \equiv \sinh \sigma_k$ with $\sigma_k = \frac{1}{2} \ln \left(\frac{\omega_1}{\omega_2} \right)$.

Our main observation is now that *the generator of mixing transformations does not leave invariant the vacuum of the fields $\phi_{1,2}(x)$, say $|0\rangle_{1,2}$, since it induces an $SU(2)$ coherent state structure in this state* [8,1]. This coherent state is the vacuum for the fields $\phi_{A,B}(x)$, which we denote by $|0\rangle_{A,B}$:

$$|0\rangle_{A,B} = G^{-1}(\theta, t) |0\rangle_{1,2} \quad (13)$$

The annihilator operators for $|0\rangle_{A,B}$ are given by $a_{k,A} \equiv G^{-1}(\theta, t) a_{k,1} G(\theta, t)$, etc.:

$$a_{k,A}(t) = \cos \theta a_{k,1} + \sin \theta \left(U_k(t) a_{k,2} + V_k(t) b_{-k,2}^\dagger \right) . \quad (14)$$

Similar expressions can be obtained for $a_{k,B}$, $b_{k,A}$ and $b_{k,B}$. We also have

$${}_{1,2}\langle 0|a_{k,A}^\dagger a_{k,A}|0\rangle_{1,2} = \sin^2 \theta |V_k|^2 = \sin^2 \theta \sinh^2 \left[\frac{1}{2} \ln \left(\frac{\omega_1}{\omega_2} \right) \right] . \quad (15)$$

Corresponding results are obtained in the case of two neutral boson fields $\phi_i(x)$, $i = 1, 2$ and their conjugate momenta $\pi_i(x) = \partial_0 \phi_i(x)$, satisfying the usual commutation relations. The non-zero commutators are:

$$[\phi_i(x), \pi_i(y)]_{t=t'} = i\delta^3(\bar{x} - \bar{y}) , \quad [a_{k,i}, a_{p,i}^\dagger] = \delta^3(\bar{k} - \bar{p}) \quad (16)$$

$$\phi_i(x) = \int \frac{d^3 \bar{k}}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\omega_i}} \left(a_{k,i} e^{-ik \cdot x} + a_{k,i}^\dagger e^{ik \cdot x} \right) \quad (17)$$

$$\pi_i(x) = \int \frac{d^3 \bar{k}}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{\omega_i}{2}} i \left(-a_{k,i} e^{-ik \cdot x} + a_{k,i}^\dagger e^{ik \cdot x} \right) \quad (18)$$

with $k \cdot x = \omega t - \bar{k} \cdot \bar{x}$. The generator of the mixing relations (corresponding to eqs. (4-5)) now is given by $G(\theta, t) = \exp[-i \theta \int d^3 \bar{x} (\pi_1(x) \phi_2(x) - \pi_2(x) \phi_1(x))]$, and again is (at finite volume) an unitary operator. By introducing the operators $S_+ \equiv -i \int d^3 \bar{x} \pi_1(x) \phi_2(x)$ and $S_- \equiv -i \int d^3 \bar{x} \pi_2(x) \phi_1(x)$, $G(\theta, t)$ can be written as $G(\theta, t) = \exp[\theta(S_+ - S_-)]$. Again, by introducing S_3 and the total "charge" S_0 as follows

$$S_3 \equiv \frac{-i}{2} \int d^3 \bar{x} (\pi_1(x) \phi_1(x) - \pi_2(x) \phi_2(x)) , \quad S_0 \equiv \frac{-i}{2} \int d^3 \bar{x} (\pi_1(x) \phi_1(x) + \pi_2(x) \phi_2(x)) \quad (19)$$

the $su(2)$ algebra is closed and by expanding the S 's operators in terms of creation and annihilation operators expressions similar to eqs.(12-15) are obtained.

For fermion field mixing we consider directly the case of neutrino field for sake of shortness. However, in this section, our considerations apply to any Dirac field. The two flavor mixing relations (for the case of three flavors see ref.[1]) are:

$$\begin{aligned}\nu_e(x) &= \nu_1(x) \cos \theta + \nu_2(x) \sin \theta \\ \nu_\mu(x) &= -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta .\end{aligned}\quad (20)$$

Here $\nu_e(x)$ and $\nu_\mu(x)$ are the (Dirac) neutrino fields with definite flavors. $\nu_1(x)$ and $\nu_2(x)$ are the (free) neutrino fields with definite masses m_1 and m_2 , respectively (we do not need to distinguish between left-handed and right-handed components):

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{k,r} [u_{k,i}^r(t) \alpha_{k,i}^r e^{i\vec{k}\cdot\vec{x}} + v_{k,i}^r(t) \beta_{k,i}^{r\dagger} e^{-i\vec{k}\cdot\vec{x}}], \quad i = 1, 2, \quad (21)$$

with $\alpha_{k,i}^r|0\rangle_{12} = \beta_{k,i}^r|0\rangle_{12} = 0$, $i = 1, 2$, $r = 1, 2$. For simplicity, the vector \vec{k} and its modulus are denoted by the same symbol. The non-zero anticommutation relations are:

$$\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(\vec{x} - \vec{y}) \delta_{\alpha\beta} \delta_{ij}, \quad \alpha, \beta = 1, \dots, 4, \quad (22)$$

$$\{\alpha_{k,i}^r, \alpha_{q,j}^{s\dagger}\} = \delta_{kq} \delta_{rs} \delta_{ij}; \quad \{\beta_{k,i}^r, \beta_{q,j}^{s\dagger}\} = \delta_{kq} \delta_{rs} \delta_{ij}, \quad i, j = 1, 2. \quad (23)$$

The orthonormality and completeness relations are the usual ones. As in the boson case, we rewrite the mixing relations (1) in the form: $\nu_e^\alpha(x) = G^{-1}(\theta, t) \nu_1^\alpha(x) G(\theta, t)$, $\nu_\mu^\alpha(x) = G^{-1}(\theta, t) \nu_2^\alpha(x) G(\theta, t)$, and the generator is $G(\theta, t) = \exp[\theta(S_+ - S_-)]$, with

$$S_+ \equiv \int d^3\vec{x} \nu_1^\dagger(x) \nu_2(x), \quad S_- \equiv \int d^3\vec{x} \nu_2^\dagger(x) \nu_1(x) = S_+^\dagger. \quad (24)$$

It is easy to see, by introducing $S_3 \equiv \frac{1}{2} \int d^3\vec{x} (\nu_1^\dagger(x) \nu_1(x) - \nu_2^\dagger(x) \nu_2(x))$, that the $su(2)$ algebra is closed: $[S_+, S_-] = 2S_3$, $[S_3, S_\pm] = \pm S_\pm$.

As in the boson case, the main point[1] is that the above generator of mixing transformations does not leave invariant the vacuum of the free fields $\nu_{1,2}$, say $|0\rangle_{1,2}$, since it induces an $SU(2)$ coherent state structure of neutrino-antineutrino pairs in this state. This coherent state is the vacuum for the fields $\nu_{e,\mu}$, which we denote by $|0\rangle_{e,\mu}$:

$$|0\rangle_{e,\mu} = G^{-1}(\theta, t) |0\rangle_{1,2}. \quad (25)$$

In the infinite volume limit $|0\rangle_{e,\mu}$ is orthogonal to $|0\rangle_{1,2}$ which exhibits the non unitary character of the mixing transformations[1].

We can then construct the Fock space for the mixed field operators which are written as:

$$\nu_e(x) = \frac{1}{\sqrt{V}} \sum_{k,r} e^{i\vec{k}\cdot\vec{x}} [u_{k,1}^r(t) \alpha_{k,e}^r(t) + v_{-k,1}^r(t) \beta_{-k,e}^{r\dagger}(t)] \quad (26)$$

$$\nu_\mu(x) = \frac{1}{\sqrt{V}} \sum_{k,r} e^{i\vec{k}\cdot\vec{x}} [u_{k,2}^r(t) \alpha_{k,\mu}^r(t) + v_{-k,2}^r(t) \beta_{-k,\mu}^{r\dagger}(t)] \quad (27)$$

where the wave functions for the massive fields have been used [1,2] and (in the reference frame $k = (0, 0, |k|)$) the creation and annihilation operators are given by:

$$\alpha_{k,e}^r(t) = \cos \theta \alpha_{k,1}^r + \sin \theta \left(U_k^*(t) \alpha_{k,2}^r + \epsilon^r V_k(t) \beta_{-k,2}^{r\dagger} \right) \quad (28)$$

$$\alpha_{k,\mu}^r(t) = \cos \theta \alpha_{k,2}^r - \sin \theta \left(U_k(t) \alpha_{k,1}^r - \epsilon^r V_k(t) \beta_{-k,1}^{r\dagger} \right) \quad (29)$$

$$\beta_{-k,e}^r(t) = \cos \theta \beta_{-k,1}^r + \sin \theta \left(U_k^*(t) \beta_{-k,2}^r - \epsilon^r V_k(t) \alpha_{k,2}^{r\dagger} \right) \quad (30)$$

$$\beta_{-k,\mu}^r(t) = \cos \theta \beta_{-k,2}^r - \sin \theta \left(U_k(t) \beta_{-k,1}^r + \epsilon^r V_k(t) \alpha_{k,1}^{r\dagger} \right) \quad (31)$$

with $\epsilon^r = (-1)^r$ and

$$V_k(t) = |V_k| e^{i(\omega_{k,2} + \omega_{k,1})t}, \quad U_k(t) = |U_k| e^{i(\omega_{k,2} - \omega_{k,1})t} \quad (32)$$

$$|U_k| = \left(\frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left(\frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left(1 + \frac{k^2}{(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)} \right) \quad (33)$$

$$|V_k| = \left(\frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left(\frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left(\frac{k}{(\omega_{k,2} + m_2)} - \frac{k}{(\omega_{k,1} + m_1)} \right) \quad (34)$$

$$|U_k|^2 + |V_k|^2 = 1 \quad (35)$$

$$|V_k|^2 = |V(k, m_1, m_2)|^2 = \frac{k^2 [(\omega_{k,2} + m_2) - (\omega_{k,1} + m_1)]^2}{4 \omega_{k,1} \omega_{k,2} (\omega_{k,1} + m_1) (\omega_{k,2} + m_2)} \quad (36)$$

where $\omega_{k,i} = \sqrt{k^2 + m_i^2}$. The number operator $N_{\sigma_l}^{k,r}$ has vacuum expectation value:

$${}_{1,2} \langle 0 | N_{\sigma_l}^{k,r} | 0 \rangle_{1,2} = \sin^2 \theta |V_k|^2, \quad \sigma = \alpha, \beta, \quad l = e, \mu, \quad (37)$$

This last equation gives the condensation density of the vacuum state $|0\rangle_{1,2}$ as a function of the mixing angle θ , of the masses m_1 and m_2 and of the momentum modulus k . Notice the difference with the usual approximation case where one puts $|0\rangle_{e,\mu} = |0\rangle_{1,2} \equiv |0\rangle$ and it is $\langle 0 | N_{\alpha_e}^{k,r} | 0 \rangle = \langle 0 | N_{\alpha_\mu}^{k,r} | 0 \rangle = 0$. Also note that ${}_{1,2} \langle 0 | N_{\sigma_l}^{k,r} | 0 \rangle_{1,2}$ plays the role of zero point contribution when considering the energy contribution of $\sigma_l^{k,r}$ particles [1].

Let us close this section with the following remarks. To be definite let us consider the fermion field case. We observe that the mixing relations (20) relate the "free" hamiltonian $H_{1,2}$ (consider only mass terms) and $H_{e,\mu}$ [6] which includes also interacting terms:

$$H_{1,2} = m_1 \bar{\nu}_1 \nu_1 + m_2 \bar{\nu}_2 \nu_2 \quad (38)$$

$$H_{e,\mu} = m_{ee} \bar{\nu}_e \nu_e + m_{\mu\mu} \bar{\nu}_\mu \nu_\mu + m_{e\mu} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e) \quad (39)$$

where $m_{ee} = m_1 \cos^2 \theta + m_2 \sin^2 \theta$, $m_{\mu\mu} = m_1 \sin^2 \theta + m_2 \cos^2 \theta$ and $m_{e\mu} = (m_2 - m_1) \sin \theta \cos \theta$. In the LSZ formalism of QFT [3,4] observables are expressed in terms of asymptotic in- (or out-) fields (also called free or physical fields) obtained by weak limit of Heisenberg or interacting fields for $t \rightarrow -(or+)\infty$. The dynamics, i.e. the Lagrangian and the resulting field equations, is given in terms of the Heisenberg fields. The meaning of weak limit is to provide a realization of the basic dynamics in terms of the asymptotic fields. Since infinitely many representations of the canonical (anti-) commutation relations exist in QFT [3,4] the weak limit is however not unique. As a consequence the realization of the basic dynamics in terms of the asymptotic fields is not unique and therefore, in order to avoid ambiguities (unitarily inequivalent representations describe physically different phases), much care is needed in the study of the mapping among Heisenberg fields and free fields (generally known as dynamical mapping or Haag expansion) [3,4].

For example, in theories with spontaneous symmetry breaking the same set of Heisenberg field equations describes the normal (symmetric) phase as well as the symmetry broken phase, according to the representation one chooses for the asymptotic fields.

Notice that in quantum mechanics no problem arises with uniqueness of the asymptotic limit since finite volume systems are considered. In such a case in fact, due to the von Neumann theorem, representations of the canonical commutation relations are each other unitary equivalent. However, in QFT where infinite number of degrees of freedom is considered the von Neumann theorem does not hold and much care is then required when considering any mapping among interacting and free fields [3,4]. We have seen in fact that, in the case of field mixing, $|0\rangle_{e,\mu}$ is orthogonal to $|0\rangle_{1,2}$ in the infinite volume limit. Field mixing relations, which can be seen as a mapping among Heisenberg fields and free fields, deserve thus a careful analysis for reasons intrinsic to the QFT structure.

3 The neutrino oscillation formula

The neutrino oscillation formula is obtained by using the mixing mappings (28-31) [1]:

$$\begin{aligned} \langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle &= \\ &= 1 - \sin^2 \theta |V_k|^2 - |U_k|^2 \sin^2 2\theta \sin^2 \left(\frac{\Delta\omega_k}{2} t \right). \end{aligned} \quad (40)$$

The number of $\alpha_\mu^{k,r}$ particles in the same state is

$$\begin{aligned} \langle \alpha_{k,e}^r(t) | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r(t) \rangle &= \\ &= |U_k|^2 \sin^2 2\theta \sin^2 \left(\frac{\Delta\omega_k}{2} t \right) + \sin^2 \theta |V_k|^2 \left(1 - \sin^2 \theta |V_k|^2 \right). \end{aligned} \quad (41)$$

The vacuum condensate contributes with the terms with $|V_k|^2$ and $|U_k|^2$ in (40) and (41). We observe that

$$\langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle + \langle \alpha_{k,e}^r(t) | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r(t) \rangle = \langle \alpha_{k,e}^r | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r \rangle + \langle \alpha_{k,e}^r | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r \rangle. \quad (42)$$

where $|\alpha_{k,e}^r\rangle = |\alpha_{k,e}^r(t=0)\rangle$, which shows the conservation of the number ($N_{\alpha_e}^{k,r} + N_{\alpha_\mu}^{k,r}$). The expectation value of this number in the state $|0\rangle_{1,2}$ is not zero due to the condensate contribution. The (approximate) relations corresponding to eqs.(40) and (41) in the conventional treatment are[5,6]:

$$\langle \alpha_{k,e}^r(t) | N_{\alpha_e}^{k,r} | \alpha_{k,e}^r(t) \rangle = 1 - \sin^2 2\theta \sin^2 \left(\frac{\Delta\omega_k}{2} t \right) \quad (43)$$

and

$$\langle \alpha_{k,e}^r(t) | N_{\alpha_\mu}^{k,r} | \alpha_{k,e}^r(t) \rangle = \sin^2 2\theta \sin^2 \left(\frac{\Delta\omega_k}{2} t \right), \quad (44)$$

respectively. The conventional (approximate) results (43) and (44) are recovered when the condensate contributions are missing (in the $|V_k| \rightarrow 0$ limit). In conclusion, in the QFT treatment we obtain corrections to the flavor oscillations which come from the condensate contributions.

The function $|V_k|^2$ has been studied in ref.[2] where the phenomenological implications of the results (40) and (41) have also been discussed. Note that $|V_k|^2$ depends on k only through its modulus and it is always in the interval $[0, \frac{1}{2}]$. It has a maximum for $k = \sqrt{m_1 m_2}$ and $|V_k|^2 = 0$ when $m_1 = m_2$. Also, $|V_k|^2 \rightarrow 0$ when $k \rightarrow \infty$.

The corrections disappear in the infinite momentum or relativistic limit $k \gg \sqrt{m_1 m_2}$ ($\sqrt{m_1 m_2}$ is the scale of the condensation density). However, for finite k , the oscillation amplitude is depressed by a factor $|U_k|^2$: this "squeezing" factor ranges from 1 to $\frac{1}{2}$ depending on k and on the masses values and thus it may have not negligible effects in experimental observations. The dependence of the flavor oscillation amplitude on the momentum could thus be tested.

To study the effects of the momentum dependence $|V_k|^2$ is written as

$$|V_k|^2 \equiv |V(p, a)|^2 = \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + a \left(\frac{p}{p^2+1} \right)^2}} \right) \quad (45)$$

$$p = \frac{k}{\sqrt{m_1 m_2}} \quad , \quad a = \frac{(\Delta m)^2}{m_1 m_2} \quad , \quad 0 \leq a < +\infty \quad , \quad (46)$$

where $\Delta m \equiv m_2 - m_1$ (we take $m_1 \leq m_2$). At $p = 1$, $|V(p, a)|^2$ reaches its maximum value $|V(1, a)|^2$, which goes asymptotically to 1/2 when $a \rightarrow \infty$.

It is useful to calculate the value of p , say p_ϵ , at which the function $|V(p, a)|^2$ becomes a fraction ϵ of its maximum value $V(1, a)$:

$$p_\epsilon = \sqrt{-c + \sqrt{c^2 - 1}} \quad , \quad c \equiv \frac{b^2(a+2) - 2}{2(b^2 - 1)} \quad , \quad b \equiv 1 - \epsilon \left(1 - \frac{2}{\sqrt{a+4}} \right) \quad . \quad (47)$$

From Tab.I-III we see that the oscillation amplitudes of neutrinos of not very large momentum may have sensible squeezing factors. We observe that large passive detectors include neutrino momentum as low as few hundreds of keV [5] and therefore deviations from the usual oscillation formula may be expected in these low momentum ranges.

An interesting case[2] occurs when one of the two masses, say m_1 , goes to zero. Then the maximum of the condensation density occurs at $k \simeq 0$; however, since $a \rightarrow \infty$ when $m_1 \rightarrow 0$, it is still possible to have sensible effects at rather "large" momenta; m_2 should be large in order to provide appreciable corrections. The situation is illustrated in Tab. III, where for the calculation we used $m_1 = 10^{-10} eV$.

Since $|U_k|^2$ has a minimum at $k = \sqrt{m_1 m_2}$ the dependence of oscillating amplitude on the momentum, if experimentally tested, may provide an indication on neutrino masses.

As we have seen, the vacuum condensate induces the correction factor; the vacuum thus acts as a "momentum (or spectrum) analyzer" for oscillating neutrinos: neutrinos with $k \gg \sqrt{m_1 m_2}$ have oscillation amplitude larger than neutrinos with $k \simeq \sqrt{m_1 m_2}$, due

to vacuum structure. This "vacuum spectral analysis" effect may sum up to other effects (such as MSW effect [9] in matter) in depressing or enhancing neutrino oscillations (see ref.[1] for a generalization of the above scheme to oscillations in matter).

In conclusion, probing the non relativistic momentum domain may provide new insights into neutrino physics.

4 Green's functions for mixed fields

In this section we present preliminary results [10] of the Green's function formalism for mixed fields in QFT. We will consider the case of (Dirac) fermion fields. As in the previous sections we consider the mixing problem only for two fields.

Let us observe that the flavor operator α_e has contributions from α_1, α_2 but *also* from the anti-particle operator β_2^\dagger (and similarly for other flavor operators in eqs.(28-31)). This additional contribution is due to the fact that the spinor wave functions for different masses are not orthogonal. In the more traditional treatment of mixing the β_2^\dagger contribution is missed since the non-orthogonality of the spinor wave functions is not considered. An important point here is time dependence of flavor operators in eqs.(28-31). The transition amplitude from a flavor eigenstate created by $\alpha_{k,e}^{r\dagger}(t')$ at time t' into the same state at time t is then calculated using the transformations (20) with the result

$$\mathcal{P}_{ee}^r(t - t', \bar{k}) = i u_{k,1}^{r\dagger} e^{i\omega_{k,1}(t-t')} \tilde{G}_{ee}^>(t - t', \bar{k}) \gamma^0 u_{k,1}^r . \quad (48)$$

Here, $\tilde{G}_{ee}^>(t - t', \bar{k})$ denotes the Fourier transform of the unordered Green function (or Wightman function) in the state that $\alpha_{k,e}^{r\dagger}(t')$ is acting upon. The Fock space vacuum state we use is $|0\rangle_{12}$, and the relative Wightman function is

$$iG_{ee}^{>\alpha\beta}(t, \bar{x}; t', \bar{y}) = {}_{1,2}\langle 0 | \nu_e^\alpha(t, \bar{x}) \bar{\nu}_e^\beta(t', \bar{y}) | 0 \rangle_{1,2} . \quad (49)$$

We obtain, for $t' = 0$,

$$\tilde{G}_{ee}^{>\alpha\beta}(t, \bar{k}) = -i \sum_r \left(\cos^2 \theta e^{-i\omega_{k,1}t} u_{k,1}^{r,\alpha} \bar{u}_{k,1}^{r,\beta} + \sin^2 \theta e^{-i\omega_{k,2}t} u_{k,2}^{r,\alpha} \bar{u}_{k,2}^{r,\beta} \right) . \quad (50)$$

The probability amplitude for the survival of the single neutrino state from initial time $t' = 0$ up to time t which follows from eq. (48), is, independent of the spin orientation,

$$\mathcal{P}_{ee}(t, \bar{k}) = \cos^2 \theta + \sin^2 \theta |U_k|^2 e^{-i(\omega_{k,2} - \omega_{k,1})t} . \quad (51)$$

The “survival” probability amplitude of an electronic neutrino state for very small times should give 1 if our definition (49) were correct: $\lim_{t \rightarrow 0^+} \mathcal{P}_{ee}(t) = 1$. One obtains instead

$$\mathcal{P}_{ee}(0^+, \bar{k}) = \cos^2 \theta + \sin^2 \theta |U_k|^2 < 1 . \quad (52)$$

This means that the vacuum state $|0\rangle_{1,2}$, which contains a pair condensate of the flavor neutrinos, cannot be used in the calculation of Green’s functions. We note that the above result for the survival probability amplitude reproduces the Pontecorvo oscillation formula in the relativistic limit $|\bar{k}| \gg \sqrt{m_1 m_2}$ where $|U_k|^2 \simeq 1$.

At time $t = 0$ the state which is annihilated by the flavor operators $\alpha_{e,\mu}(0)$ and $\beta_{e,\mu}(0)$, is the “flavor vacuum”: $|0\rangle_{e,\mu} \equiv G^{-1}(\theta, 0)|0\rangle_{1,2}$. The correct definition of the Wightman function for ν_e is then

$$iG_{ee}^{>\alpha\beta}(t, \bar{x}; 0, \bar{y}) = {}_{e,\mu}\langle 0 | \nu_e^\alpha(t, \bar{x}) \bar{\nu}_e^\beta(0, \bar{y}) | 0 \rangle_{e,\mu} . \quad (53)$$

to be compared with eq.(49) where $|0\rangle_{1,2}$ is used. An important point in the definition (53) arises from the fact that one may not directly compare states at different times. Since the generator of the mixing transformations is time dependent (see Sec. 2), the relative orientation of the ν_1 - ν_2 Fock space and the flavor Fock space is twisted with time. Therefore the comparison of states at different times necessitates a *parallel transport* of these states to a common point of reference. The definition (53) includes this concept of parallel transport, and indeed the fixation of a zero point is a gauge fixing also in the usual way of QFT: the richness of the geometric structure of mixing transformations, i.e., gauge fixing, parallel transport and geometric phases will be considered in a separate publication.

Also, we note that up to a phase factor the same function as (53) is obtained if on *both sides* the flavor vacuum state is replaced by the state $|0_t\rangle_{e,\mu} = G^{-1}(\theta, t)|0\rangle_{1,2}$. This is in agreement with our comment on the parallel transport: one may compare the two state vectors at either end of the time trajectory. However, a different result is obtained when replacing the flavor vacuum state on both sides of eq. (53) by $|0_\tau\rangle_{e,\mu} = G^{-1}(\theta, \tau)|0\rangle_{1,2}$ with arbitrary τ . The reason is that the product $\nu_e^\dagger(t, \bar{x})|0_\tau\rangle_{e,\mu}$ cannot be interpreted as a single electron neutrino created at time τ .

Eq. (53) is conveniently expressed in terms of anticommutators at different times as

$$\begin{aligned} \tilde{G}_{ee}^{>\alpha\beta}(t, \bar{k}) &= -i \sum_r \left[u_{k,1}^{r,\alpha} e^{-i\omega_{k,1}t} \bar{u}_{k,1}^{r,\beta} \left\{ \alpha_{k,e}^r(t), \alpha_{k,e}^{r\dagger}(0) \right\} \right. \\ &\quad \left. + v_{-k,1}^{r,\alpha} e^{i\omega_{k,1}t} \bar{u}_{k,1}^{r,\beta} \left\{ \beta_{-k,e}^{r\dagger}(t), \alpha_{k,e}^{r\dagger}(0) \right\} \right] . \end{aligned} \quad (54)$$

This structure shows that our definition of probability amplitudes singles out one anti-commutator by time:

$$\begin{aligned}\mathcal{P}_{ee}^r(t, \bar{k}) &= i u_{k,1}^{r\dagger} e^{i\omega_{k,1}t} \tilde{G}_{ee}^>(t, \bar{k}) \gamma^0 u_{k,1}^r = \left\{ \alpha_{k,e}^r(t), \alpha_{k,e}^{r\dagger}(0) \right\} \\ &= \cos^2\theta + \sin^2\theta \left[|U_k|^2 e^{-i(\omega_{k,2}-\omega_{k,1})t} + |V_k|^2 e^{i(\omega_{k,2}+\omega_{k,1})t} \right]\end{aligned}\quad (55)$$

$$\begin{aligned}\mathcal{P}_{\bar{e}e}^r(t, \bar{k}) &= i v_{-k,1}^{r\dagger} e^{-i\omega_{k,1}t} \tilde{G}_{ee}^>(t, \bar{k}) \gamma^0 u_{k,1}^r = \left\{ \beta_{-k,e}^{r\dagger}(t), \alpha_{k,e}^{r\dagger}(0) \right\} \\ &= \epsilon^r |U_k| |V_k| \sin^2\theta \left[e^{i(\omega_{k,2}-\omega_{k,1})t} - e^{-i(\omega_{k,2}+\omega_{k,1})t} \right]\end{aligned}\quad (56)$$

Note that the probability amplitude is now normalized correctly: $\lim_{t \rightarrow 0^+} \mathcal{P}_{ee}(t, \bar{k}) = 1$. For completeness we define also the “mixed” Green function

$$iG_{\mu e}^{>\alpha\beta}(t, \bar{x}; 0, \bar{y}) =_{e,\mu} \langle 0 | \nu_\mu^\alpha(t, \bar{x}) \bar{\nu}_e^\beta(0, \bar{y}) | 0 \rangle_{e,\mu} . \quad (57)$$

$$\begin{aligned}\mathcal{P}_{\mu e}^r(t, \bar{k}) &= i u_{k,2}^{r\dagger} e^{i\omega_{k,2}t} \tilde{G}_{\mu e}^>(t, \bar{k}) \gamma^0 u_{k,1}^r = \left\{ \alpha_{k,\mu}^r(t), \alpha_{k,e}^{r\dagger}(0) \right\} \\ &= |U_k| \cos\theta \sin\theta \left[1 - e^{i(\omega_{k,2}-\omega_{k,1})t} \right]\end{aligned}\quad (58)$$

$$\begin{aligned}\mathcal{P}_{\bar{\mu}e}^r(t, \bar{k}) &= i v_{-k,2}^{r\dagger} e^{-i\omega_{k,2}t} \tilde{G}_{\mu e}^>(t, \bar{k}) \gamma^0 u_{k,1}^r = \left\{ \beta_{-k,\mu}^{r\dagger}(t), \alpha_{k,e}^{r\dagger}(0) \right\} \\ &= \epsilon^r |V_k| \cos\theta \sin\theta \left[1 - e^{-i(\omega_{k,2}+\omega_{k,1})t} \right] .\end{aligned}\quad (59)$$

All other anticommutators with $\alpha_e^\dagger(0)$ vanish. The conservation of the total probability then requires that

$$\left| \mathcal{P}_{ee}(t, \bar{k}) \right|^2 + \left| \mathcal{P}_{\bar{e}e}(t, \bar{k}) \right|^2 + \left| \mathcal{P}_{\mu e}(t, \bar{k}) \right|^2 + \left| \mathcal{P}_{\bar{\mu}e}(t, \bar{k}) \right|^2 = 1 , \quad (60)$$

which is of course satisfied by our calculation.

Further work on the Green’s functions formalism is in progress.

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Table I: The values of $\sqrt{m_1 m_2}$ and of a for given values of m_1 and m_2 .

	$m_1 (eV)$	$m_2 (KeV)$	$\sqrt{m_1 m_2} (KeV)$	a
A	5	250	1.12	$\sim 5 \cdot 10^4$
B	2.5	250	0.79	$\sim 1 \cdot 10^5$
C	5	200	1	$\sim 4 \cdot 10^4$
D	1	100	0.32	$\sim 1 \cdot 10^5$
E	0.5	50	0.15	$\sim 1 \cdot 10^5$
F	0.5	1	0.02	$\sim 2 \cdot 10^3$

Table II: $|U(p_\epsilon, a)|^2$ vs. k_ϵ .

	$ U(1, a) ^2$	$k_1 (KeV)$	$ U(p_{1/2}, a) ^2$	$k_{1/2} (KeV)$	$ U(p_{1/10}, a) ^2$	$k_{1/10} (KeV)$
A	$\simeq 0.5$	1.12	$\simeq 0.75$	$\simeq 146$	$\simeq 0.95$	$\simeq 519$
B	$\simeq 0.5$	0.79	$\simeq 0.75$	$\simeq 145$	$\simeq 0.95$	$\simeq 518$
C	$\simeq 0.5$	1	$\simeq 0.75$	$\simeq 117$	$\simeq 0.95$	$\simeq 415$
D	$\simeq 0.5$	0.32	$\simeq 0.75$	$\simeq 58$	$\simeq 0.95$	$\simeq 206$
E	$\simeq 0.5$	0.16	$\simeq 0.75$	$\simeq 29$	$\simeq 0.95$	$\simeq 104$
F	$\simeq 0.5$	0.02	$\simeq 0.75$	$\simeq 0.6$	$\simeq 0.95$	$\simeq 2$

Table III: $|U(p_\epsilon, a)|^2$ vs. k_ϵ for $m_1 \simeq 0$ and different values of m_2 .

$m_1 (eV)$	$m_2 (KeV)$	$ U(p_{1/2}, a) ^2$	$k_{1/2} (KeV)$	$ U(p_{1/10}, a) ^2$	$k_{1/10} (KeV)$
$\simeq 0$	250	$\simeq 0.75$	$\simeq 144$	$\simeq 0.95$	$\simeq 516$
$\simeq 0$	200	$\simeq 0.75$	$\simeq 115$	$\simeq 0.95$	$\simeq 413$
$\simeq 0$	100	$\simeq 0.75$	$\simeq 57$	$\simeq 0.95$	$\simeq 206$
$\simeq 0$	50	$\simeq 0.75$	$\simeq 29$	$\simeq 0.95$	$\simeq 103$